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PROPER G-SPACES

WAYNE E. DYDO

Introduction

By a G-space we will mean a completely regular topological space X on which a locally compact topological group G acts continuously on the left. If G is a Lie group, X is a differentiable (C^∞) manifold, and the action is differentiable, then we call X a *differentiable G-space*. We will assume that the reader is familiar with the concepts of *Cartan G-space*, *proper G-space*, and *slice* defined by Palais in [5].

Among the many results of [5] is the fact that if X is a separable metrizable proper G-space with G a Lie group, then each orbit of X is closed, each isotropy group is compact, and there is a metric defined on X with respect to which G acts on X as a group of isometries.

In § 1 we prove the following converse of this result.

Theorem A. *Let X be a connected locally compact metric G-space with G a second countable Lie group acting effectively on X as a group of isometries. If there is a p in X with Gp closed and G_p compact, then X is a proper G-space.*

For a G-space X on which G is a Lie group acting freely, the triple $X(X/G, G)$ is a principal fibre bundle if and only if X is a proper G-space. This result appears in § 4 of [5].

The differentiable version of this result is also true. Specifically, we prove the following theorem in § 2.

Theorem B, *Let X be a differentiable G-space with G acting freely on X and* $\dim G > 0$. *Then X is a proper G-space if and only if $X(X/G, G)$ is a differentiable principal fibre bundle.*

In § 3 we show that the parallelizability theorem of L. Markus (see [4]) is a special case of Theorem B.

0. Notation

Let X be a G-space. For p in X and g in G, let gp denote the image of the pair (g, p) under the action of G. Let $Gp = \{gp \mid g \in G\}$, $G_p = \{g \in G \mid gp = p\}$.

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Call *Gp the orbit of X through p* and G_p the *isotropy group of G at p*. The *orbit space* provided with the quotient topology is denoted by X/G.

1. Proof of Theorem A

In this section X will denote a connected locally compact metric space, G a second countable Lie group, and $I(X)$ the isometry group of X provided with the compact-open topology.

The proof of the following lemma may be found on pp. 47–49 of [3].

Lemma 1. *Let $\{\varphi_n\}$ be a sequence in $I(X)$, and p a point in X. Suppose $\{\varphi_n(p)\}$ converges in X. Then there is a subsequence of $\{\varphi_n\}$ converging in $I(X)$.*

Lemma 2. *Assume that G acts effectively on X as a group of isometries, so that we identify G with a subgroup of $I(X)$. If there is a p in X with Gp closed in X and G_p closed in $I(X)$, then G is closed in $I(X)$ and all orbits of X are closed. Moreover X/G is metrizable.*

Proof. Give $I(X)/G_p$ (left coset space) the quotient topology. Then G_p closed in $I(X)$ implies that $I(X)/G_p$ is Hausdorff. Regard G/G_p as a subset of $I(X)/G_p$. The manifold topology of G contains the subspace topology of G inherited from $I(X)$. It follows that the manifold topology of G/G_p contains the subspace topology of G/G_p inherited from $I(X)/G_p$.

Gp closed implies that $s_p\colon G/G_p \to Gp$ defined by $s_p(gG_p) = pg$ is a homeomorphism.

Let $\{g_n\}$ be a sequence in G, and φ in $I(X)$ with $g_n \to \varphi$ in $I(X)$. Then $g_n p \to \varphi(p)$ in X. Let $\varphi(p) = gp$ for some g in G. Then $s_p^{-1}(g_n p) = g_n G_p \to gG_p = s_p^{-1}(gp)$ in G/Gp with the subspace topology. Therefore $g_n G_p \to gG_p$ in $I(X)/G_p$. But $g_n \to \varphi$ implies $g_n G_p \to \varphi G_p$ in $I(X)/G_p$. Therefore $I(X)/G_p$ Hausdorff implies that $gG_p = \varphi G_p$. In particular φ is in G. Thus G is closed in $I(X)$. This immediately implies that all the orbits are closed.

Let d be the metric of X, and $\pi\colon X \to X|G$ the projection. For p and q in X set $\bar{d}(\pi(p), \pi(q)) = d(Gp, Gq)$. One easily verifies that $\bar{d}$ is a metric for X/G, which induces the quotient topology. q.e.d.

Assume that X is a G-space. For p in X set $J(p) = \{q \in X \mid \exists$ sequences $\{p_n\}$ and $\{g_n\}$ in X and G respectively such that $p_n \to p$, $g_n p_n \to q$, and $\{g_n\}$ has no convergent subsequence in $G\}$. Call $J(p)$ the *prolongational limit set of p*. We use $J(p)$ to characterize Cartan G-spaces.

Lemma 3. *X is a Cartan G-space if and only if $p \notin J(p)$ for all p in X.*

Proof. Assume that X is a Cartan G-space. Let $p \in J(p)$ with $p_n \to p$, $g_n p_n \to p$, and $\{g_n\}$ having no convergent subsequence in G. Let U be an open neighborhood of p with $(U, U) = \{g \in G \mid gU \cap U \neq \phi\}$ relatively compact. For large n, p_n and $g_n p_n$ are in U so that g_n is in (U, U) and hence $\{g_n\}$ contains a convergent subsequence, a contradiction.

Conversely assume that $p \notin J(p)$ for all p in X. For p in M suppose G_p is not compact. Then there is a sequence $\{g_n\}$ in G_p having no convergent subsequence in G_p and hence in G since G_p is closed in G. But this implies that $p \in J(p)$ by letting $p_n = p$, a contradiction. Thus for all p in X, G_p is compact.

If X is not a Cartan G-space, then there are a p in M and a sequence $\{U_n\}$ of open neighborhoods of p with $U_{n+1} \subset U_n$, (U_n, U_n) not relatively compact, and $\bigcap_{n=1}^{\infty} U_n = \{p\}$. Choose an open neighborhood U of e in G (where e is the identity) so that $G_p \subset U$ and U is relatively compact. Then there is a g_n in $(U_n, U_n) - U$. g_n in (U_n, U_n) implies that there is a p_n in U_n such that $g_n p_n$ is in U_n. $\bigcap_{n=1}^{\infty} U_n = \{p\}$ implies that $p_n \to p$ and $g_n p_n \to p$. Since $p \notin J(p)$, $\{g_n\}$ has a convergent subsequence, say $\{g_{n_k}\}$ with $g_{n_k} \to g$. Then $g_{n_k} p_{n_k} \to p$, $p_{n_k} \to p$, $g_{n_k} \to g$ imply that $p = gp$. Thus g is in G_p, and hence g_{n_k} is in U for large n_k, a contradiction.

Proof of Theorem A. Identify G with a subgroup of $I(X)$. Let T_m be the manifold topology of G, and T_s the subspace topology inherited from $I(X)$. Then the identity $\iota: (G, T_m) \to (G, T_s)$ is a continuous homomorphism. Thus by [2, Corollary 3.3, p. 111] ι is also open, so that $T_s = T_m$. Hence G_p compact implies that G_p is closed in $I(X)$. By Lemma 2, G is closed in $I(X)$ and X/G is Hausdorff.

Suppose that for q in X, $q \in J(q)$ with $q_n \to q$, $g_n q_n \to q$ and $\{g_n\}$ having no convergent subsequence in G. Let d be the metric of X. Then $d(q, g_n q) \leq d(q, g_n q_n) + d(g_n q_n, g_n q) = d(q, g_n q_n) + d(q_n, q) \to 0$. Thus $g_n q \to q$. By Lemma 1, $\{g_n\}$ contains a subsequence convergent in $I(X)$ and hence in G since G is closed and $T_s = T_m$. This contradicts the assumption on $\{g_n\}$. Thus $q \notin J(q)$ for all q in X.

Hence Theorem A follows from Lemma 3 and Theorem 1.2.9 of [5].

2. Proof of Theorem B

Proof. Assume that X is a proper G-space. Since G acts freely on X, there exist complete vector fields V_i on X, $i = 1, \cdots, m = \dim G$, such that for all p in M, $\{V_i(p)\}$ is a basis for the tangent space $T_p(Gp)$. Therefore given p in M we can find a coordinate chart $(U, y = y_1, \cdots, y_n))$, $n = \dim X$, about p with $y(p) = 0$ and $V_i(p) = \partial/\partial y_i(p), i = 1, \cdots, m$. Let $S_p^* = \{q \in U \mid y_i(q) = 0, i = 1, \cdots, m\}$. Then S_p^* is a submanifold of X, $p \in S_p^*$, and by making U smaller if necessary we may assume that for all q in S_p^*, $T_q(X) = T_q(Gq) \oplus T_q(S_p^*)$. By § 2.2 and Proposition 2.1.7 of [5] there exists an open set S_p in S_p^* such that S_p is a slice at p. It is easily verified that the map $\alpha_p: G \times S_p \to GS_p$ defined by $\alpha_p(g, q) = gq$ is a diffeomorphism.

Let $\pi: X \to X/G$ be the projection. Then π is open. For each p in X, choose S_p and α_p as above. It is readily verified that $\pi|_{S_p}$ maps S_p homeomorphically onto the open set $\pi(S_p)$, and if we set $\psi_p = \pi|_{S_p}{}^{-1}$, then for p and q in X with $\pi(S_p) \cap \pi(S_q) \neq \phi$, $\psi_q \circ \psi_p^{-1}: \psi_p(\pi(S_p) \cap \pi(S_q)) \to \psi_q(\pi(S_p) \cap \pi(S_q))$ is a diffeomorphism. Since $\{\pi(S_p)\}$ covers X/G, by choosing as coordinate charts parirs of the form (U, φ) where (V, ψ) is a coordinate chart in some S_p,

$U = \pi(V)$ and $\varphi = \psi \circ \psi_p$ we have a C^∞ atlas on X/G such that each ψ_p is a diffeomorphism. By Theorem 1.2.9 of [5] X/G is Hausdorff. Thus X/G is a differentiable manifold. It is easily verified that π is C^∞ and $\pi^{-1}(\pi(S_p)) \approx \pi(S_p) \times G$ by $gq \to (\pi(q), g)$ where $g \in G$ and $q \in S_p$. Hence $X(X/G, G)$ is a differentiable principal fibre bundle.

Conversely, if $X(X/G, G)$ is a differentiable principal fibre bundle with G acting on X on the left, then X/G is Hausdorff; and if p is in X, choose U to be an open neighborhood of $\pi(p)$ in X/G with $\beta\colon \pi^{-1}(U) \approx U \times G$. Let $\beta(p) = (\pi(p), g)$ and $S_p = \beta^{-1}(U \times \{g\})$. It is readily verified that S_p is a slice at p, and from Theorems 1.2.9 and 2.3.3 of [5] it follows that X is a proper G-space.

Corollary. *Let X be a paracompact differentiable manifold, and R^m a Euclidean m-space. Then X is a differentiable proper R^m-space if and only if X is diffeomorphic to a product $N \times R^m$.*

Proof. If X is a differentiable proper R^m-space, then from Proposition 1.1.4 of [5] R^m acts freely on X. By Theorem B, $X(X/R^m, R^m)$ is a differentiable principal fibre bundle. By Theorem 4.3.1 of [5] R^m acts on X as a group of isometries with respect to a Riemannian metric. From Lemma 2 it follows that X/R^m is paracompact. Thus the corollary follows from the following theorem whose proof may be found on pp. 58–59 of [3].

Theorem. *If $X(X/R^m, R^m)$ is a differentiable principal fibre bundle with X/R^m paracompact, then $X(X/R^m, R^m)$ admits a cross section. If s is a cross section, then $f\colon X/R^m \times R^m \to X$ defined by $f(y, t) = ts(y)$ is a diffeomorphism.*

The converse is obvious.

Corollary. *Let X be a Riemannian manifold, and V a complete Killing vector field on X. Assume that the action of R $(= R^1)$ on X induced by the one-parameter group of V is free, and that one integral curve of V is closed. Then X is diffeomorphic to a product $N \times R$ by a diffeomorphism f with $f_*(V) = \partial/\partial x$ where $\{x\}$ is the usual coordinate system on R.*

Proof. From Theorem A it follows that X is a proper differentiable R-space where the action is given by the one-parameter group of V. The above corollary yields the existence of an $f\colon X \approx X/R \times R$ where f^{-1} is of from $(y, t) \to ts(y)$ for s a cross section of $X(X/R, R)$. An easy computation shows that $f_*^{-1}(\partial/\partial x) = V$.

3. Parallelizability

In this section X will be a connected paracompact differentiable manifold, and V a complete vector field on X. Via the one-parameter group of V, X is a differentiable R-space denoted by $X_{(V)}$.

In [4] Markus defined the concepts of a *completely unstable* complete vector field and a complete vector field *without separatricies*. From Theorem 2 of [4] and Theorem 1.2.9 of [5] it follows that V is completely unstable if and only if $X_{(V)}$ is a Cartan R-space, and that V is completely unstable without separatricies if and only if $X_{(V)}$ is a proper R-space (see [1] for details). Thus the first corollary to Theorem B and the proof of the second corollary to Theorem B give the parallelizability theorem of Markus (Theorem 4 of [4]).

Theorem. *Assume that V is completely unstable and without separatricies. Then X/R is a differentiable manifold, and there is an $f: X \approx X/R \times R$ such that $f_*(V) = \partial/\partial x$ where $\{x\}$ is the usual coordinate system on R.*

Remark. Let g be a Riemannian metric on X, and $L_V g$ the Lie derivative of g with respect to V. Assume that $L_V g$ is again a Riemannian metric and that V never vanishes. Then $X_{(V)}$ is a proper R-space. If X vanishes at some point p, then p is unique, $X - \{p\}_{(V)}$ is a proper R-space, and X is diffeomorphic to a Euclidean space. For details see [1].

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